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J. Phys. A: Math. Gen. **34** (2001) 4459–4474 www.iop.org/Journals/ja PII: S0305-4470(01)16268-2

Algebraic Bethe ansatz for integrable extended Hubbard models arising from supersymmetric group solutions

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Received 7 August 2000, in final form 27 March 2001

Abstract

Integrable extended Hubbard models arising from symmetric group solutions are examined in the framework of the graded quantum inverse scattering method. The Bethe ansatz equations for all these models are derived by using the algebraic Bethe ansatz method.

PACS number: 7127

1. Introduction

The study of exactly solvable models of strongly correlated electrons is important for understanding fundamental aspects of statistical mechanics. It is relevant to many realistic physical systems such as high-T_c superconductors. The quantum inverse scattering method (QISM) has been applied to solve various strongly correlated electron models [1]. Two models which have attracted a great deal of attention are the Hubbard model, which is derived from an R-matrix with non-additive spectral parameter, and which was investigated by Shastry [2], and its strong coupling limit the t–J model. Essler and Korepin [3] established the integrability of the t–J model by obtaining an infinite set of conserved quantities, and studied the model in the framework of the graded QISM. An integrable anisotropic (q-deformed) supersymmetric t–J model was proposed by Foerster and Karowski [4]. Woynarovich [5] applied the finite-size correction of the ground-state energy to obtain the low-lying gapless excitation spectrum around the ground state. Frahm and Korepin [6] obtained the critical exponents for various correlation functions. Furthermore, the Bethe ansatz solution and conformal properties of the q-deformed t–J model were studied by Bariev *et al* [7].

By considering other representations of quantum superalgebras, new integrable, strongly correlated electron models of interest, such as the supersymmetric Essler–Korepin–Schoutens (EKS) extended Hubbard model [8], have been proposed. This $gl(2|2)$ supersymmetric model contains the supersymmetric t–J model as a submodel and can be interpreted as the Hubbard model plus moderate nearest-neighbour interactions. The complete solution of the EKS model by the algebraic Bethe ansatz has been obtained [9]. The mathematical issue of the completeness of the solution has been settled [10], and the physics content of the solution, low-lying excitations in particular, has been studied [11]. Another non-standard integrable model of strongly correlated electrons is the Bariev chain [12], which has an R-matrix with non-additive spectral parameter as investigated by Zhou [13]. A generalization of the Hubbard model with Lie superalgebra $gl(2|1)$ supersymmetry, the supersymmetric U model, was discovered in [14], and has been investigated by several groups [15–18]. An extension of this model, a q-deformed version with quantum superalgebra $U_q(gl(2|1))$ supersymmetry, was also proposed [19, 20]. Thermodynamic properties of the EKS model and the supersymmetric U model have been studied using Wiener–Hopf techniques and the critical exponents of correlation functions by using methods of conformal field theory [19]. Recently, the eight-state supersymmetric U model of strongly correlated electrons with the Lie superalgebra $gl(3|1)$ symmetry, and the two-parameter (q-deformed) eight-state supersymmetric fermion model with quantum superalgebra $U_q(gl(3|1))$ symmetry, were introduced [21, 22].

Dolcini and Montorsi [23] introduced integrable extended Hubbard Hamiltonians from symmetric group solutions. One of the aims of this paper is to show the solution of these models via the QISM. The most general form of the extended Hubbard model invariant under spin-flip and conserving the total number of electrons and magnetization, first considered in [24], is described by the Hamiltonian

$$\begin{aligned} H = \mu_{\mathrm{e}} \sum_{j,\sigma} n_{j,\sigma} &- \sum_{\langle j,k\rangle,\sigma} [t - X(n_{j,-\sigma} + n_{k,-\sigma}) + \tilde{X} n_{j,-\sigma} n_{k,-\sigma}] c_{j,\sigma}^{\dagger} c_{k,\sigma} + U \sum_{j} n_{j,\uparrow} n_{j,\downarrow} \\ &+ \frac{V}{2} \sum_{\langle j,k\rangle} n_j n_k + \frac{W}{2} \sum_{\langle j,k\rangle,\sigma,\sigma'} c_{j,\sigma}^{\dagger} c_{k,\sigma'}^{\dagger} c_{j,\sigma'} c_{k,\sigma} + Y \sum_{\langle j,k\rangle} c_{j,\uparrow}^{\dagger} c_{j,\downarrow}^{\dagger} c_{k,\downarrow} c_{k,\uparrow} \\ &+ P \sum_{\langle j,k\rangle} n_{j,\uparrow} n_{j,\downarrow} n_k + \frac{Q}{2} \sum_{\langle j,k\rangle} n_{j,\uparrow} n_{j,\downarrow} n_{k,\uparrow} n_{k,\downarrow} \end{aligned} \qquad (1)$$

where μ_{e} is the chemical potential. Here, electrons on a lattice are described by canonical Ferimi operators $c_{j,\alpha}$ and $c_{j,\alpha}^{\dagger}$ satisfying the anticommutation relations given by $\{c_{i,\alpha}^{\dagger}, c_{j,\beta}\} = \delta_{ij}\delta_{\alpha\beta}$, where $i, j, = 1, 2, \ldots,$ and $\alpha, \beta = \uparrow, \downarrow$. In addition, $n_{j,\alpha} = c_{j,\alpha}^{\dagger} c_{j,\alpha}$, $n_j = n_{j,\downarrow} + n_{j,\uparrow}$. In (1) the term t represents the band energy of the electrons, while the subsequent terms describe their Coulomb interaction energy in a narrow band approximation: U parametrizes the on-site diagonal interaction, V the neighbouring-site–charge interaction, X the bond–charge interaction, W the exchange term, and Y the pair-hopping term. Moreover, additional many-body coupling terms have been included in agreement with [24]: $\tilde{X}$ correlates hopping with on-site occupation number, and P and Q describe three- and four-electron interactions.

As was shown in [23], the integrability of the model lies in the fact that there exists a solution of the Yang–Baxter equation which takes the form

$$\check{R}(u) = 1 + u\Pi$$

where u is the spectral parameter and

$$\Pi = \begin{pmatrix}
\sigma_{11}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_{12}^{\mathrm{d}} & 0 & 0 & \sigma_{12}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{13}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & \sigma_{13}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma_{14}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{14}^{\mathrm{o}} & 0 & 0 & 0 \\
0 & \sigma_{12}^{\mathrm{o}} & 0 & 0 & \sigma_{21}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{22}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{23}^{\mathrm{d}} & 0 & 0 & \sigma_{23}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{24}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & \sigma_{24}^{\mathrm{o}} & 0 & 0 \\
0 & 0 & \sigma_{13}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & \sigma_{31}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{23}^{\mathrm{o}} & 0 & 0 & \sigma_{32}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{33}^{\mathrm{d}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{34}^{\mathrm{d}} & 0 & 0 & \sigma_{34}^{\mathrm{o}} & 0 \\
0 & 0 & 0 & \sigma_{14}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{41}^{\mathrm{d}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{24}^{\mathrm{o}} & 0 & 0 & 0 & 0 & 0 & \sigma_{42}^{\mathrm{d}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{34}^{\mathrm{o}} & 0 & 0 & \sigma_{43}^{\mathrm{d}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{44}^{\mathrm{d}}
\end{pmatrix}. \qquad (2)$$

Indeed, after making the assignment

$$|1\rangle = |0\rangle \qquad |2\rangle = |\downarrow\uparrow\rangle_j = c^\dagger_{j,\downarrow} c^\dagger_{j,\uparrow}|0\rangle$$
$$|3\rangle = |\uparrow\rangle_j = c^\dagger_{j,\uparrow}|0\rangle \qquad |4\rangle = |\downarrow\rangle_j = c^\dagger_{j,\downarrow}|0\rangle \tag{3}$$

one can show that H can be expressed as the sum over a graded *generalized permutator*

$$H = \sum_{j=1}^{L} \Pi_{j,j+1} \tag{4}$$

where the operator $\Pi_{j,j+1}$ permutes the four possible configurations (3) between the sites j and $j+1$; namely

$$\Pi = \sigma^{\rm d}_{ik}(e_{ii} \otimes_s e_{kk}) + \sigma^{\rm o}_{ik}(e_{ik} \otimes_s e_{ki})$$

where $\sigma^{\rm d}_{ik}(e_{ii} \otimes e_{kk})$ are diagonal terms and $\sigma^{\rm o}_{ik}(e_{ik} \otimes e_{ki})$ are off-diagonal terms. At this point we would like to mention that the construction here bears some resemblance to those by Alcaraz and Bariev [25] and Maassarani [26] for solutions of the Yang–Baxter equation based on representations of the Hecke algebra.

It is clear that this form of interaction conserves the individual numbers $N_\uparrow$ and $N_\downarrow$ of electrons with spin up and spin down, respectively, and the numbers $N_{\rm l}$ and $N_{\rm h}$ of doubly occupied (local electron pairs) and empty sites (holes). We will choose the following conventions throughout this paper:

$N_\uparrow$ = number of single electrons with spin up
$N_\downarrow$ = number of single electrons with spin down
$N_{\rm e} = N_\uparrow + N_\downarrow$ = number of single electrons
$N_{\rm l}$ = number of local electron pairs
$N_{\rm h}$ = number of holes
$N_{\rm b} = N_{\rm h} + N_{\rm l}$ = number of 'bosons'.

To be specific, we give the relations between $\sigma^{\rm d}_{i,k}$, $\sigma^{\rm o}_{i,k}$ and parameters in the Hamiltonian (1):

$$\sigma^{\rm d}_{11} = c \qquad \sigma^{\rm d}_{22} = c + Q + U + 4P + 2\mu_{\rm e} + 4V - 2W \qquad \sigma^{\rm d}_{33} = \sigma^{\rm d}_{44} = c + V - W + \mu_{\rm e}$$
$$\sigma^{\rm d}_{12} = \sigma^{\rm d}_{21} = c + \mu_{\rm e} + \frac{U}{2} \qquad \sigma^{\rm d}_{13} = \sigma^{\rm d}_{14} = \sigma^{\rm d}_{31} = \sigma^{\rm d}_{41} = c + \frac{\mu_{\rm e}}{2}$$
$$\sigma^{\rm d}_{23} = \sigma^{\rm d}_{24} = \sigma^{\rm d}_{32} = \sigma^{\rm d}_{42} = c + P + \frac{3\mu_{\rm e}}{2} + \frac{U}{2} + 2V - W \qquad \sigma^{\rm d}_{34} = \sigma^{\rm d}_{43} = c + V + \mu_{\rm e}$$
$$\sigma^{\rm o}_{12} = Y \qquad \sigma^{\rm o}_{13} = \sigma^{\rm o}_{14} = -t \qquad \sigma^{\rm o}_{23} = \sigma^{\rm o}_{24} = -t + \tilde{X} \qquad \sigma^{\rm o}_{34} = -W.$$

It turns out that actually there are 96 different possible choices of values of the physical parameters in (1). They can be cast into six groups as (2) shows; see table 1 in which there is the restriction $s_i = \pm 1$, $i = 1, \ldots, 5$.

We now construct the eigenstates of the Hamiltonians of the one-dimensional model in the six groups of table 1, using the QISM. We use the R-matrix satisfying the Yang–Baxter equation and introduce an L-operator constructed directly from the R-matrix of the twisted representation. The quantum Yang–Baxter equation can be written as the operator equation

$$\check{R}(\lambda - \mu) L_j(\lambda) \otimes L_j(\mu) = L_j(\mu) \otimes L_j(\lambda) \check{R}(\lambda - \mu). \tag{5}$$

Here, $\otimes$ denotes the graded tensor product defined by

$$(A \otimes B)_{ij,kl} = (-1)^{(\epsilon_i + \epsilon_j)\epsilon_k} A_{ij} B_{kl}$$

Table 1.

	$H_1(s_1,\ldots,s_5)$	$H_2(s_1,\ldots,s_5)$	$H_3(s_1,s_2,s_3)$	$H_4(s_1,s_2,s_3)$	$H_5(s_1,s_2,s_3)$	$H_6(s_1,s_2,s_3)$
t	1	1	1	1	1	0
X	1	1	1	1	1	0
$\tilde{X}$	$1+s_2$	$1+s_2$	$1+s_2$	$1+s_2$	1	1
U	$2s_1$	$2s_1$	$4s_1$	$4s_1$	$2s_1$	$-2s_1$
V	s_1	s_1+s_4	s_1	s_1+s_3	s_1+s_3	0
W	s_4	0	s_3	0	0	0
Y	s_3	s_3	0	0	s_2	s_2
P	s_4-s_1	$-s_1-2s_4$	s_3-2s_1	$-2(s_1+s_3)$	$-(s_1+s_3)$	0
Q	$-2s_4+s_1+s_5$	$4s_4+s_1+s_5$	$4s_1-2s_3$	$4(s_1+s_3)$	s_1+s_3	s_1+s_3
μ_e	$-2s_1$	$-2s_1$	$-2s_1$	$-2s_1$	$-2s_1$	0
c	s_1	s_1	s_1	s_1	s_1	s_1

where $\epsilon_i \in \boldsymbol{Z}_2$ denotes the grading of the index i. We chose to adopt the bosonic, bosonic, fermionic and fermionic (BBFF) grading $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = \epsilon_4 = 1$ on the indices labelling the basis vectors.

We now proceed to establish the relation between the Hamiltonian (1) and the transfer matrix $\tau(\lambda)$, which is the supertrace of the monodromy matrix $T(\lambda)$ defined by

$$T(\lambda) = L_L(\lambda)L_{L-1}\cdots L_1(\lambda).$$

From (5) it follows that

$$\check{R}(\lambda-\mu)T(\lambda)\otimes T(\mu) = T(\mu)\otimes T(\lambda)\check{R}(\lambda-\mu). \tag{6}$$

Thus we have

$$[\tau(\lambda), \tau(\mu)] = 0$$

and so the $\tau(\lambda)$ form a one-parameter family of commuting operators. The transfer matrices may be taken as integrals of the motion, and so we obtain an infinite number of higher conservation laws of the model.

2. Algebraic Bethe ansatz for group 1

Having recalled the quantum integrability of the models, let us use the nested algebraic Bethe ansatz method to find the eigenvalues of the transfer matrices. As will be shown, the Bethe ansatz solutions for the six different groups take on differing forms. In particular, the number of levels of Bethe ansatz nestings ranges from 0 to 2. Moreover, some cases do not admit a unique reference state. In such an instance, we are forced to use a subspace of reference states to perform the calculations. This type of procedure was first investigated by Abad and Ríos [27] for the case of alternating $su(3)$ representations and we will adopt this formalism where necessary.

We start from the first group with BBFF grading. The explicit form of the R-matrix is

$$\check{R}(u) = 1 + u\Pi$$

$$
= \begin{pmatrix}
1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & s_3u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 \\
0 & s_3u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+s_5u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 \\
0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-s_4u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -s_4u & 0 \\
0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_4u & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-s_4u
\end{pmatrix}.
$$

To utilize the framework of the QISM, we write down the L-operator

$$L_j(u) = \frac{1}{1+s_1u} P\check{R}(u) \tag{7}$$

where P is the graded permutation operator

$$P = \sum_{ij} (-1)^{[j]} e_{ij} \otimes e_{ji}. \tag{8}$$

We choose the local vacuum state as $|0\rangle_j = (0, 0, 0, 1)^t$. Acting the L-operator on this local vacuum state, we have

$$L_j(u)|0\rangle_j = \begin{pmatrix} 1 & * & * & * \\ 0 & s_3a(u) & 0 & 0 \\ 0 & 0 & -a(u) & 0 \\ 0 & 0 & 0 & -a(u) \end{pmatrix} |0\rangle_j \tag{9}$$

with $a(u) = u/(1+s_1u)$. Define the vacuum state as $|0\rangle = \otimes_{j=1}^{L} |0\rangle_j$. Using the standard QISM, we represent the monodromy matrix in the following way:

$$T(u) = L_L(u)L_{L-1}(u)\cdots L_1(u) \equiv \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) & D_{13}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) & D_{23}(u) \\ C_3(u) & D_{31}(u) & D_{32}(u) & D_{33}(u) \end{pmatrix}. \tag{10}$$

The transfer matrix with periodic boundary conditions is thus written explicitly as

$$\tau(u) = A(u) + D_{11}(u) - D_{22}(u) - D_{33}(u). \tag{11}$$

The action of the monodromy matrix on the pseudo-vacuum state is

$$\begin{aligned}
&A(u)|0\rangle = |0\rangle \qquad D_{11}(u)|0\rangle = [s_3a(u)]^L|0\rangle \\
&D_{22}(u)|0\rangle = D_{33}(u)|0\rangle = [-a(u)]^L|0\rangle \\
&B_k(u)|0\rangle \neq 0 \qquad C_k(u)|0\rangle = 0 \\
&D_{kl}(u)|0\rangle = 0 \qquad (k \neq l, \quad k, l = 1, 2, 3).
\end{aligned} \tag{12}$$

Substituting (10) into the Yang–Baxter algebra (6) we may deduce the following commutation relations:

$$D_{ab}(\mu)B_c(\lambda) = S_a \left[(-1)^{\epsilon_a\epsilon_p} \frac{r(\mu-\lambda)_{pb}^{dc}}{a(\mu-\lambda)} B_p(\lambda)D_{ad}(\mu) - (-1)^{\epsilon_a\epsilon_b} \frac{b(\mu-\lambda)}{a(\mu-\lambda)} B_b(\mu)D_{ac}(\lambda) \right]$$

$$A(\mu)B_c(\lambda) = S_c \left[\frac{1}{a(\lambda-\mu)} B_c(\lambda)A(\mu) - \frac{b(\lambda-\mu)}{a(\lambda-\mu)} B_c(\mu)A(\lambda) \right]$$

$$B_{a_1}(\lambda)B_{a_2}(\mu) = r(\lambda-\mu)_{b_2a_1}^{b_1a_2} B_{b_2}(\mu)B_{b_1}(\lambda)$$

where

$$r(u) = b(u) I^{(1)ab}_{cd} + a(u)\Pi^{(1)ab}_{cd}$$
$$a(u) = \frac{u}{1+s_1 u} \qquad b(u) = \frac{1}{1+s_1 u}.$$

Here, $\Pi^{(1)ab}_{cd}$ is the 9×9 submatrix of Π,

$$\Pi^{(1)} = \begin{pmatrix} s_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_4 & 0 \\ 0 & 0 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 \end{pmatrix}$$

corresponding to the grading $\epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1$ and $S_1 = \frac{1}{s_3}, S_2 = -1, S_3 = -1$. We assume the eigenvectors of the transfer matrix to be

$$B_{a_1}(\lambda_1)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)|0\rangle\, F^{a_n\cdots a_1}$$

where $F^{a_1\cdots a_n}$ is a function of the spectral parameters λ_j. Acting the transfer matrix (11) on the above vector, we have

$$\begin{aligned}\tau(u)&B_{a_1}(\lambda_1)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)|0\rangle\, F^{a_n\cdots a_1}\\ &= [A(u)+D_{11}(u)-D_{22}(u)-D_{33}(u)]B_{a_1}(\lambda_1)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)|0\rangle\, F^{a_n\cdots a_1}\\ &= s_3^{-N_1}(-1)^{N_e}\prod_{j=1}^{n}\frac{1}{a(\lambda_j-u)}B_{a_1}(\lambda_1)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)|0\rangle\, F^{a_n\cdots a_1}\\ &\quad +[a(u)]^L s_3^{L-n}(-1)^{N_e+N_\downarrow}\prod_{j=1}^{n}\frac{1}{a(\lambda_j-u)}\tau^{(1)}(u,\{\lambda_k\})^{a_n\cdots a_1}_{b_n\cdots b_1}\\ &\quad \times B_{b_1}(\lambda_1)B_{b_2}(\lambda_2)\cdots B_{b_n}(\lambda_n)|0\rangle\, F^{a_n\cdots a_1} + \text{u.t.}\end{aligned}$$

where u.t. denotes unwanted terms, and $\tau^{(1)}(u)$ is the nested transfer matrix. Denote the eigenvalues of $\tau(u)$ and $\tau^{(1)}(u)$ by $\Lambda(u)$ and $\Lambda^{(1)}(u)$. We have

$$\Lambda(u) = s_3^{-N_1}(-1)^{N_e}\prod_{j=1}^{n}\frac{1}{a(\lambda_j-u)} + [a(u)]^L s_3^{L-n}(-1)^{N_e+N_\downarrow}\prod_{j=1}^{n}\frac{1}{a(\lambda_j-u)}\Lambda^{(1)}(u).$$

In order to cancel the unwanted terms, we need the following Bethe ansatz equations:

$$\Lambda^{(1)}(\lambda_k) = s_3^{-N_1}(-1)^{N_e}\left[s_3^{L-n}(-1)^{N_e+N_\downarrow}\right]^{-1}[a(\lambda_k)]^{(-L)}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{a(\lambda_k-\lambda_j)}{a(\lambda_j-\lambda_k)}.$$

The nested transfer matrix is written as the supertrace on the auxiliary space for the reduced monodromy matrix which satisfies the Yang–Baxter relation.

$$\tau^{(1)}(u,\{\lambda_k\}) = \operatorname{str}\left[\operatorname{diag}\left(\frac{1}{s_3},-1,-1\right)L^{(1)}_n(u-\lambda_n)L^{(1)}_{n-1}(u-\lambda_{n-1})\cdots L^{(1)}_1(u-\lambda_1)\right] \quad (13)$$
$$r(\lambda-\mu)T^{(1)}_n(\lambda)\otimes T^{(1)}_n(\mu) = T^{(1)}_n(\mu)\otimes T^{(1)}_n(\lambda)r(\lambda-\mu).$$

If we write

$$T^{(1)}_n(u) = L^{(1)}_n(u)L^{(1)}_{n-1}(u)\cdots L^{(1)}_1(u) \equiv \begin{pmatrix} A^{(1)}(u) & B^{(1)}_1(u) & B^{(1)}_2(u)\\ C^{(1)}_1(u) & D^{(1)}_{11}(u) & D^{(1)}_{12}(u)\\ C^{(1)}_2(u) & D^{(1)}_{21}(u) & D^{(1)}_{22}(u)\end{pmatrix} \quad (14)$$

the $L^{(1)}$-operator is

$$L_j^{(1)}(u) = P^{(1)} r_j(u)$$

where $P^{(1)}$ is the 9×9 permutation operator

$$P^{(1)} = \sum_{ij} (-1)^{\epsilon_j} e_{ij} \otimes e_{ji} \tag{15}$$

corresponding to the grading $\epsilon_1 = 0, \epsilon_3 = \epsilon_3 = 1$. Now (13) and $r(u)$ imply that

$$D_{ab}^{(1)}(\mu) B_c^{(1)}(\lambda) = \frac{1}{s_2} \left[(-1)^{\epsilon_a \epsilon_p} \frac{r^{(1)}(\mu - \lambda)_{pb}^{dc}}{a^{(1)}(\mu - \lambda)} B_p^{(1)}(\lambda) D_{ad}^{(1)}(\mu) \right.$$
$$\left. - (-1)^{\epsilon_a \epsilon_b} \frac{b^{(1)}(\mu - \lambda)}{a^{(1)}(\mu - \lambda)} B_b(\mu) D_{ac}(\lambda) \right]$$
$$A^{(1)}(\mu) B_c^{(1)}(\lambda) = \frac{1}{s_2} \left[\frac{1}{a^{(1)}(\lambda - \mu)} B_c^{(1)}(\lambda) A^{(1)}(\mu) - \frac{b^{(1)}(\lambda - \mu)}{a^{(1)}(\lambda - \mu)} B_c^{(1)}(\mu) A^{(1)}(\lambda) \right]$$
$$B_{a_1}^{(1)}(\lambda) B_{a_2}^{(1)}(\mu) = r^{(1)}(\lambda - \mu)_{b_2 a_1}^{b_1 a_2} B_{b_2}^{(1)}(\mu) B_{b_1}^{(1)}(\lambda).$$

Here the values 1, 2 are both fermionic ($\epsilon_1 = 1 = \epsilon_2$). The R-matrix $r^{(1)}(\mu)$ is

$$r^{(1)}(u)_{cd}^{ab} = b^{(1)}(u) I^{(2)}{}_{cd}^{ab} + a^{(1)}(u) \Pi^{(2)}{}_{cd}^{ab}$$
$$a^{(1)}(u) = \frac{u}{1 + s_5 u} \qquad b^{(1)}(u) = \frac{1}{1 + s_5 u}.$$

In the above, $\Pi^{(2)}{}_{cd}^{ab}$ is a 4×4 submatrix of $\Pi^{(1)}$

$$\Pi^{(2)} = \begin{pmatrix} -s_4 & 0 & 0 & 0 \\ 0 & 0 & -s_4 & 0 \\ 0 & -s_4 & 0 & 0 \\ 0 & 0 & 0 & -s_4 \end{pmatrix} \tag{16}$$

corresponding to the grading $\epsilon_1 = \epsilon_2 = 1$. As the reference state for the first nesting we choose $|0\rangle_k^{(1)} = (1, 0, 0)^t$, $|0\rangle^{(1)} = \otimes_{k=1}^n |0\rangle_k^{(1)}$ as the pseudo-vacuum. We find that

$$A^{(1)}(u)|0\rangle^{(1)} = |0\rangle^{(1)}$$
$$D_{11}^{(1)}(u)|0\rangle^{(1)} = D_{22}^{(1)}(u)|0\rangle^{(1)} = \prod_{j=1}^{n} s_2 a^{(1)}(u - \lambda_j)|0\rangle^{(1)}$$

and due to $\tau^{(1)}(u) = \frac{1}{s_3} A^{(1)}(u) + D_{11}^{(1)}(u) + D_{22}^{(1)}(u)$ we obtain the eigenvalue

$$\Lambda^{(1)}(u, \{\lambda_k\}) = \frac{1}{s_3} \prod_{j=1}^{n_1} \frac{1}{s_2 a^{(1)}(\lambda_j^{(1)} - u)}$$
$$+ \prod_{j=1}^{n_1} \frac{1}{s_2 a^{(1)}(u - \lambda_j^{(1)})} \prod_{k=1}^{n} s_2 a^{(1)}(u - \lambda_k) \Lambda^{(2)}(u, \{\lambda_m^{(1)}\})$$

provided the parameters $\{\lambda_m^{(1)}\}$ satisfy

$$\Lambda^{(2)}(\lambda_m^{(1)}) = \frac{1}{s_3} \prod_{\substack{l=1 \\ l \neq m}}^{n_1} \frac{a^{(1)}(\lambda_m^{(1)} - \lambda_l^{(1)})}{a^{(1)}(\lambda_l^{(1)} - \lambda_m^{(1)})} \prod_{k=1}^{n} \frac{1}{s_2 a^{(1)}(\lambda_m^{(1)} - \lambda_k)}.$$

The transfer matrix of the second nesting is written as

$$\tau^{(2)}(u, \{\lambda_m^{(1)}\}) = \mathrm{str}\left[L_{n_1}^{(2)}(u - \lambda_{n_1}^{(1)}) L_{n_1-1}^{(2)}(u - \lambda_{n_1-1}^{(1)}) \cdots L_1^{(2)}(u - \lambda_1^{(1)}) \right]$$

where

$$L_k^{(2)}(u) = \begin{pmatrix} a^{(2)}(u) - b^{(2)}(u)e_k^{11} & -b^{(2)}(u)e_k^{21} \\ -b^{(2)}(u)e_k^{12} & a^{(2)} - b^{(2)}(u)e_k^{22} \end{pmatrix}. \qquad (17)$$

From the Yang–Baxter relation for $\tau^{(2)}(u)$ one can derive the following intertwining relation:

$$r^{(1)}(\lambda-\mu)T_{n_1}^{(2)}(\lambda)\otimes T_{n_1}^{(2)}(\mu) = T_{n_1}^{(2)}(\mu)\otimes T_{n_1}^{(2)}(\lambda)r^{(1)}(\lambda-\mu). \qquad (18)$$

The components of (18) needed for the construction of an algebraic Bethe ansatz are

$$\begin{aligned}
D^{(2)}(\mu)B^{(2)}(\lambda) &= \frac{1}{s_4}\left[\frac{1}{a^{(2)}(\lambda-\mu)}B^{(2)}(\lambda)D^{(2)}(\mu) + \frac{b^{(2)}(\mu-\lambda)}{a^{(2)}(\mu-\lambda)}B^{(2)}(\mu)D^{(2)}(\lambda)\right] \\
A^{(2)}(\mu)B^{(2)}(\lambda) &= \frac{1}{s_4}\left[\frac{1}{a^{(2)}(\mu-\lambda)}B^{(2)}(\lambda)A^{(2)}(\mu) + \frac{b^{(2)}(\lambda-\mu)}{a^{(2)}(\lambda-\mu)}B^{(2)}(\mu)A^{(2)}(\lambda)\right] \qquad (19) \\
B^{(2)}(\lambda)B^{(2)}(\mu) &= B^{(2)}(\mu)B^{(2)}(\lambda)
\end{aligned}$$

where

$$a^{(2)}(u) = \frac{u}{1+s_4u} \qquad b^{(2)}(u) = \frac{1}{1+s_4u}.$$

For the reference state for the second nesting we pick $|0\rangle_k^{(2)} = (1,0)^t$, $|0\rangle^{(2)} = \otimes_{k=1}^{n_1}|0\rangle_k^{(2)}$. From the action of the nested monodromy matrix

$$T_{n_1}^{(2)}(u) = L_{n_1}^{(2)}(u)L_{n_1-1}^{(2)}(u)\cdots L_1^{(2)}(u) \equiv \begin{pmatrix} A^{(2)}(u) & B^{(2)}(u) \\ C^{(2)}(u) & D^{(2)}(u) \end{pmatrix}$$

we find that

$$A^{(2)}(u)|0\rangle^{(2)} = \prod_{j=1}^{n_1}\frac{a^{(2)}(u-\lambda_j^{(1)})}{a^{(2)}(\lambda_j^{(1)}-u)}|0\rangle^{(1)} \qquad D^{(2)}(u)|0\rangle^{(1)} = \prod_{j=1}^{n_1}s_4a^{(2)}(u-\lambda_j^{(1)})|0\rangle^{(1)}$$

due to $\tau^{(2)}(u) = -A^{(2)}(u) - D^{(2)}(u)$. Thus

$$\begin{aligned}
\Lambda^{(2)}(u,\{\lambda_m^{(1)}\}) = -\Bigg[&\prod_{j=1}^{n_2}\frac{1}{s_4a^{(2)}(u-\lambda_j^{(2)})}\prod_{m=1}^{n_1}\frac{a^{(2)}(u-\lambda_m^{(1)})}{a^{(2)}(\lambda_m^{(1)}-u)} \\
&+\prod_{j=1}^{n_2}\frac{1}{s_4a^{(2)}(\lambda_j^{(2)}-u)}\prod_{m=1}^{n_1}s_4a^{(2)}(u-\lambda_m^{(1)})\Bigg]
\end{aligned}$$

under the condition that the spectral parameters $\{\lambda_p^{(2)}\}$ are solutions to the Bethe ansatz equation

$$\prod_{\substack{j=1\\j\neq p}}^{n_2}\frac{a^{(2)}(\lambda_j^{(2)}-\lambda_p^{(2)})}{a^{(2)}(\lambda_p^{(2)}-\lambda_j^{(2)})} = \prod_{k=1}^{n_1}s_4a^{(2)}(\lambda_k^{(1)}-\lambda_p^{(2)}).$$

We have now obtained the complete set of nested Bethe ansatz equations, which read

$$\left(\frac{1+s_1\lambda_k}{\lambda_k}\right)^L = s_3^{N_1-1}(-1)^{-N_e}s_3^{L-n}(-1)^{N_e+N_\downarrow}\prod_{j=1}^{N_e}\frac{s_5(\lambda_j^{(1)}-\lambda_k)+1}{s_2(\lambda_j^{(1)}-\lambda_k)}\prod_{\substack{l=1\\l\neq k}}^{N_e+N_1}\frac{s_1(\lambda_k-\lambda_l)+1}{s_1(\lambda_k-\lambda_l)-1}$$

$$\begin{aligned}
\prod_{j=1}^{N_e+N_1}\frac{s_5(\lambda_m^{(1)}-\lambda_k)+1}{s_2(\lambda_m^{(1)}-\lambda_k)} &= s_3\prod_{\substack{l=1\\l\neq m}}^{N_e}\frac{s_5(\lambda_m^{(1)}-\lambda_l^{(1)})+1}{s_5(\lambda_m^{(1)}-\lambda_l^{(1)})-1}\,\frac{s_4(\lambda_m^{(1)}-\lambda_l^{(1)})-1}{s_4(\lambda_m^{(1)}-\lambda_l^{(1)})+1} \\
&\quad\times\prod_{j=1}^{N_\downarrow}\frac{s_4(\lambda_m^{(1)}-\lambda_j^{(2)})+1}{s_4(\lambda_m^{(1)}-\lambda_j^{(2)})}
\end{aligned}$$

$$\prod_{\substack{j=1\\j\neq p}}^{N_\downarrow}\frac{s_4(\lambda_j^{(2)}-\lambda_p^{(2)})-1}{s_4(\lambda_j^{(2)}-\lambda_p^{(2)})+1} = \prod_{k=1}^{N_e}\frac{s_4(\lambda_k^{(1)}-\lambda_p^{(2)})}{s_4(\lambda_k^{(1)}-\lambda_p^{(2)})+1}.$$

Here we have used $n = N_e + N_l = N_\downarrow + N_\uparrow + N_l, n_1 = N_e, n_2 = N_\downarrow$. The corresponding energy eigenvalue E of the model is given by

$$E = \sum_{j=1}^{N_e+N_l} \frac{1}{\lambda_j(1+s_1\lambda_j)} - L. \tag{20}$$

The BBFF grading solution of the above models closely follows the solution of the EKS model in [9–11].

3. Algebraic Bethe ansatz for group 2

The algebraic Bethe ansatz calculations for this group proceed in exactly the same manner as group 1 up to the introduction of the matrix $\Pi^{(1)}$, which for this case reads

$$\Pi^{(1)} = \begin{pmatrix} s_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_4 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 \end{pmatrix}.$$

Following the calculation along the same lines as the previous section, we find that the matrix $r^{(1)}(u)$ appearing in equations (19) is of the form

$$r^{(1)}(u)_{aa}^{bb} = (b^{(1)}(u) + s_4 a^{(1)}(u)) I^{(2)}{}_{aa}^{bb} = \frac{1+s_4 u}{1+s_5 u} I^{(2)}{}_{aa}^{bb}.$$

Here, $I^{(2)}{}_{aa}^{bb}$ is the 4×4 identity matrix. For the reference state of the first nesting we choose the state $|0\rangle_k^{(1)} = (1, 0, 0)^t$, $|0\rangle^{(1)} = \otimes_{k=1}^n |0\rangle_k^{(1)}$ as the pseudo-vacuum and then find that

$$A^{(1)}(u)|0\rangle^{(1)} = |0\rangle^{(1)}$$

$$D_{11}^{(1)}(u)|0\rangle^{(1)} = D_{22}^{(1)}(u)|0\rangle^{(1)} = \prod_{j=1}^{n} s_2 a^{(1)}(u-\lambda_j)|0\rangle^{(1)}.$$

Due to $\tau^{(1)}(u) = \frac{1}{s_3} A^{(1)}(u) + D_{11}^{(1)}(u) + D_{22}^{(1)}(u)$, we obtain the eigenvalue

$$\Lambda^{(1)}(u, \{\lambda_k\}) = \frac{1}{s_3} \prod_{j=1}^{n_1} \frac{1}{s_2 a^{(1)}(\lambda_j^{(1)} - u)} + \prod_{j=1}^{n_1} \frac{1}{s_2 a^{(1)}(u - \lambda_j^{(1)})} \prod_{k=1}^{n} s_2 a^{(1)}(u - \lambda_k)$$

under the condition that the spectral parameters $\{\lambda_m^{(1)}\}$ are solutions of the Bethe ansatz equation

$$\prod_{\substack{l=1 \\ l\neq m}}^{n_1} \frac{a^{(1)}(\lambda_m^{(1)} - \lambda_l^{(1)})}{a^{(1)}(\lambda_l^{(1)} - \lambda_m^{(1)})} = s_3 \prod_{k=1}^{n} s_2 a^{(1)}(\lambda_m^{(1)} - \lambda_k).$$

For the full solution, we have the nested Bethe ansatz equations

$$\left(\frac{1+s_1\lambda_k}{\lambda_k}\right)^L = s_3^{N_l-1}(-1)^{-N_e} s_3^{L-n}(-1)^{N_e+N_\downarrow} \times \prod_{j=1}^{N_e} \frac{s_5(\lambda_j^{(1)} - \lambda_k) + 1}{s_2(\lambda_j^{(1)} - \lambda_k)} \prod_{\substack{l=1 \\ l\neq k}}^{N_e+N_l} \frac{s_1(\lambda_k - \lambda_l) + 1}{s_1(\lambda_k - \lambda_l) - 1}$$

$$\prod_{j=1}^{N_e+N_l} \frac{s_5(\lambda_m^{(1)} - \lambda_k) + 1}{s_2(\lambda_m^{(1)} - \lambda_k)} = \frac{1}{s_3} \prod_{\substack{l=1 \\ l\neq m}}^{N_e} \frac{s_5(\lambda_m^{(1)} - \lambda_l^{(1)}) + 1}{s_5(\lambda_m^{(1)} - \lambda_l^{(1)}) - 1}$$

where $n,\ n_1,\ n_2$ have the same meaning as previously. In addition, the energy expression (20) applies here.

4. Algebraic Bethe ansatz for group 3

We now consider the case of the algebraic Bethe ansatz for group 3. As we will see here, the procedure is fundamentally different from the preceding cases in that we are required to work with a subspace of reference states for the first level of the algebraic Bethe ansatz. The methodology we employ follows that proposed by Abad and Ríos [27].

In the case of group 3, the R-matrix reads

$$\check{R}(u) = 1 + u\Pi$$
$$= \begin{pmatrix}
1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 \\
0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-s_3u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -s_3u & 0 \\
0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_3u & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-s_3u
\end{pmatrix} \tag{21}$$

and we again express the L-operator as

$$L_j(u) = \frac{1}{1+s_1u} P\check{R}(u).$$

If we choose the local vacuum state as $|0\rangle_j = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha, \beta, 0, 0)^t$, and act the L-operator on this local vacuum state, we have

$$L_j(u)|0\rangle_j = \begin{pmatrix} e_{11} & e_{21} & * & * \\ e_{12} & e_{22} & * & * \\ 0 & 0 & (s_2e_{22} - e_{11})a(u) & 0 \\ 0 & 0 & 0 & (s_2e_{22} - e_{11})a(u) \end{pmatrix} |0\rangle_j. \tag{22}$$

Define the vacuum state as $|0\rangle = \otimes_{j=1}^{L}|0\rangle_j$ and represent the monodromy matrix as

$$T(u) = L_L(u)L_{L-1}(u)\cdots L_1(u) \equiv \begin{pmatrix} A_{11}(u) & A_{12}(u) & B_{11}(u) & B_{12}(u) \\ A_{21}(u) & A_{22}(u) & B_{21}(u) & B_{22}(u) \\ C_{11}(u) & C_{12}(u) & D_{11}(u) & D_{12}(u) \\ C_{21}(u) & C_{22}(u) & D_{21}(u) & D_{22}(u) \end{pmatrix}. \tag{23}$$

The transfer matrix is thus written explicitly as

$$\tau(u) = A_{11}(u) + A_{22}(u) - D_{11}(u) - D_{22}(u).$$

The action of the monodromy matrix on the vacuum state is

$$[A_{11}(u) + A_{22}(u)]|0\rangle = \mathrm{tr}_0[P_{L0}P_{L-1,0}\cdots P_{10}]|0\rangle$$
$$D_{11}(u)|0\rangle = D_{22}(u)|0\rangle = [a(u)]^L(-1)^{N_e}s_2^{N_1}$$
$$B_{ik}(u)|0\rangle \neq 0 \qquad C_{ik}(u)|0\rangle = 0 \qquad (i, k = 1, 2)$$

where P_{j0} is the permutation operator for two-dimensional spaces (corresponding to the indices 1 and 2). Substituting (23) into the Yang–Baxter algebra (6), we may deduce the following commutation relations:

$$\begin{aligned} D_{ac}(\mu)B_{bd}(\lambda) &= S_b\Bigg[(-1)^{\epsilon_a\epsilon_b}\frac{r(\mu-\lambda)^{d'd}_{c'c}}{a(\mu-\lambda)}B_{ac'}(\lambda)D_{bd'}(\mu) \\ &\quad -(-1)^{\epsilon_a\epsilon_b}\frac{b(\mu-\lambda)}{a(\mu-\lambda)}B_{bc}(\mu)D_{ad}(\lambda)\Bigg] \\ A_{ac}(\mu)B_{bd}(\lambda) &= S_c\left[\frac{1}{a(\lambda-\mu)}B_{ad}(\lambda)A_{bc}(\mu)-\frac{b(\lambda-\mu)}{a(\lambda-\ mu)}B_{ad}(\mu)A_{bc}(\lambda)\right] \\ B_{ac}(\lambda)B_{bd}(\mu) &= r(\lambda-\mu)^{d'd}_{c'c}B_{ac'}(\mu)B_{bd'}(\lambda) \end{aligned} \tag{24}$$

where

$$r(u)^{ab}_{cd} = b(u)I^{(2)ab}_{\ \ cd} + a(u)\Pi^{(2)ab}_{\ \ cd}.$$

Here, $\Pi^{(2)ab}_{\ \ cd} = s_3P^{(2)}$ with permutation matrix $P^{(2)} = -\sum_{ij} e_{ij}\otimes e_{ji}$ corresponding to the grading $\epsilon_1 = \epsilon_2 = 1$ and $S_1 = -1$, $S_2 = \frac{1}{s_2}$. Denote the eigenvalues of $\tau(u)$ and $\tau^{(1)}$ by $\Lambda(u)$ and $\Lambda^{(1)}(u)$. We now have

$$\Lambda(u) = G\cdot\prod_{j=1}^{n_1}\frac{1}{a(\lambda_j-u)} + [a(u)]^L\prod_{j=1}^{n_1}\frac{1}{a(u-\lambda_j)}\Lambda^{(1)}(u).$$

Here, $G = \mathrm{tr}[\mathrm{diag}(-1,\frac{1}{s_2})P_{L0}P_{L-1,0}\cdots P_{10}]$, and the parameters $\{\lambda_k\}$ are required to satisfy the Bethe ansatz equations

$$\Lambda^{(1)}(\lambda_k) = [a(\lambda_k)]^{(-L)}\cdot G\cdot\prod_{\substack{j=1\\ j\neq k}}^{n_1}\frac{a(\lambda_k-\lambda_j)}{a(\lambda_j-\lambda_k)}.$$

The nested transfer matrix is written as the supertrace on the auxiliary space for the reduced monodromy matrix which satisfies the Yang–Baxter relation, i.e.

$$\begin{aligned} &\tau^{(1)}(u,\{\lambda_k\}) = \mathrm{str}\left[\mathrm{diag}\left(-1,\frac{1}{s_2}\right)L^{(1)}_{n_1}(u-\lambda_{n_1})L^{(1)}_{n_1-1}(u-\lambda_{n_1-1})\cdots L^{(1)}_1(u-\lambda_1)\right] \\ &r(\lambda-\mu)T^{(1)}_{n_1}(\lambda)\otimes T^{(1)}_{n_1}(\mu) = T^{(1)}_{n_1}(\mu)\otimes T^{(1)}_{n_1}(\lambda)r(\lambda-\mu). \end{aligned} \tag{25}$$

The components of (25) needed for the construction of an algebraic Bethe ansatz are

$$\begin{aligned} D^{(1)}(\mu)B^{(1)}(\lambda) &= \frac{1}{s_3}\left[\frac{1}{a^{(3)}(\lambda-\mu)}B^{(1)}(\lambda)D^{(1)}(\mu)+\frac{b^{(3)}(\mu-\lambda)}{a^{(3)}(\mu-\lambda)}B^{(1)}(\mu)D^{(1)}(\lambda)\right] \\ A^{(1)}(\mu)B^{(1)}(\lambda) &= \frac{1}{s_3}\left[\frac{1}{a^{(3)}(\mu-\lambda)}B^{(1)}(\lambda)A^{(1)}(\mu)+\frac{b^{(3)}(\lambda-\mu)}{a^{(3)}(\lambda-\mu)}B^{(1)}(\mu)A^{(1)}(\lambda)\right] \\ B^{(1)}(\lambda)B^{(1)}(\mu) &= B^{(1)}(\mu)B^{(1)}(\lambda) \end{aligned}$$

where

$$a^{(3)}(u) = \frac{u}{1+s_3u} \qquad b^{(3)}(u) = \frac{1}{1+s_3u}.$$

As the reference state for the second nesting we take $|0\rangle^{(1)}_k = (1,0)^t$, $|0\rangle^{(1)} = \otimes^{n_1}_{k=1}|0\rangle^{(2)}_k$. From the action of the nested monodromy matrix

$$T^{(1)}_{n_1}(u) = L^{(1)}_{n_1}(u)L^{(1)}_{n_1-1}(u)\cdots L^{(1)}_1(u) \equiv \begin{pmatrix} A^{(1)}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(1)}(u)\end{pmatrix}$$

we find that

$$A^{(1)}(u)|0\rangle^{(1)} = \prod_{j=1}^{n_1} \frac{a^{(3)}(u-\lambda_j)}{a^{(3)}(\lambda_j - u)}|0\rangle^{(1)} \qquad D^{(1)}(u)|0\rangle^{(1)} = \prod_{j=1}^{n_1} s_3 a^{(3)}(u-\lambda_j)|0\rangle^{(1)}$$

and due to $\tau^{(1)}(u) = A^{(1)}(u) - \frac{1}{s_2}D^{(1)}(u)$ we have

$$\Lambda^{(1)}(u,\{\lambda_k\}) = \prod_{j=1}^{n_2} \frac{1}{s_3 a^{(3)}(u-\lambda_j^{(1)})} \prod_{k=1}^{n_1} \frac{a^{(3)}(u-\lambda_k)}{a^{(3)}(\lambda_k - u)} - \frac{1}{s_2} \times \prod_{j=1}^{n_2} \frac{1}{s_3 a^{(3)}(\lambda_j^{(1)} - u)} \prod_{k=1}^{n_1} s_3 a^{(3)}(u-\lambda_k)$$

under the condition that the spectral parameters $\{\lambda_m^{(1)}\}$ are solutions to the Bethe ansatz equation

$$\prod_{\substack{j=1\\ j\neq m}}^{n_2} \frac{a^{(3)}(\lambda_j^{(1)} - \lambda_m^{(1)})}{a^{(3)}(\lambda_m^{(1)} - \lambda_j^{(1)})} = -\frac{1}{s_2}\prod_{k=1}^{n_1} s_3 a^{(3)}(\lambda_k - \lambda_m^{(1)}).$$

Now we obtain the complete set of nested Bethe ansatz equations reading

$$\left(\frac{1+s_1\lambda_k}{\lambda_k}\right)^L = -\frac{1}{G}\cdot \prod_{\substack{j=1\\ j\neq k}}^{N_e} \frac{s_1(\lambda_k-\lambda_j)-1}{s_1(\lambda_k-\lambda_j)+1}\frac{s_1(\lambda_k-\lambda_1)-1}{s_1(\lambda_k-\lambda_1)+1}\prod_{j=1}^{N_\downarrow}\frac{s_3(\lambda_j^{(1)}-\lambda_k)-1}{s_3(\lambda_j^{(1)}-\lambda_k)}$$

$$\prod_{\substack{j=1\\ j\neq p}}^{N_\downarrow} \frac{s_3(\lambda_j^{(1)}-\lambda_p^{(1)})-1}{s_3(\lambda_j^{(1)}-\lambda_p^{(1)})+1} = -\frac{1}{s_2}\prod_{k=1}^{N_e}\frac{s_3(\lambda_k-\lambda_p^{(1)})}{s_3(\lambda_k-\lambda_p^{(1)})+1}.$$

The energy expression for this model reads the same as in the previous cases (20).

5. Algebraic Bethe ansatz for group 4

Just as the calculations for the cases of groups 1 and 2 follow along similar lines, we find an analogous situation occurring with groups 3 and 4. For group 4, we have the R-matrix

$$\check{R}(u) = 1 + u\Pi$$

$$= \begin{pmatrix}
1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0\\
0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u
\end{pmatrix}. \tag{26}$$

The calculations of the algebraic Bethe ansatz proceed in exactly the same manner as the group 3 case except now we find that in (24) we have

$$r(u)_{aa}^{bb} = (b(u) + s_3 a(u)) I^{(2)}{}_{aa}^{bb} = \frac{1+s_3u}{1+s_1u} I^{(2)}{}_{aa}^{bb}.$$

For the eigenvalues $\Lambda(u)$ of $\tau(u)$ we obtain the expression

$$\Lambda(u) = G \cdot \prod_{j=1}^{n_1} \frac{1}{a(\lambda_j - u)} + [a(u)]^L \prod_{j=1}^{n_1} \frac{1}{a(u - \lambda_j)}$$

where G is defined as before and $\{\lambda_k\}$ are subject to the Bethe ansatz equations

$$\left(\frac{1 + s_1\lambda_k}{\lambda_k}\right)^L = G \cdot \prod_{\substack{j=1 \\ j\neq k}}^{N_e} \frac{s_1(\lambda_k - \lambda_j) + 1}{s_1(\lambda_k - \lambda_j) - 1}.$$

Again, the energies are given by (20).

6. Algebraic Bethe ansatz for group 5

For group 5, the R-matrix is given by

$$\check{R}(u) = 1 + u\Pi$$
$$= \begin{pmatrix}
1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 \\
0 & s_2u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_3u
\end{pmatrix}. \tag{27}$$

We choose the local vacuum state as $|0\rangle_j = (1, 0, 0, 0)^t$. Acting the L-operator on this local vacuum state, we have

$$L_j(u)|0\rangle_j = \begin{pmatrix} 1 & * & * & * \\ 0 & s_2a(u) & 0 & 0 \\ 0 & 0 & -a(u) & 0 \\ 0 & 0 & 0 & -a(u) \end{pmatrix} |0\rangle_j. \tag{28}$$

Define the vacuum state as $|0\rangle = \otimes_{j=1}^{L} |0\rangle_j$. The monodromy matrix is represented as

$$T(u) = L_L(u)L_{L-1}(u)\cdots L_1(u) \equiv \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_2(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) & D_{13}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) & D_{23}(u) \\ C_3(u) & D_{31}(u) & D_{32}(u) & D_{33}(u) \end{pmatrix} \tag{29}$$

and so the transfer matrix is explicitly

$$\tau(u) = A(u) + D_{11}(u) - D_{22}(u) - D_{33}(u).$$

The action of the monodromy matrix on the vacuum state is

$$\begin{aligned} &A(u)|0\rangle = |0\rangle \qquad D_{11}(u) = [s_2a(u)]^L|0\rangle \qquad D_{22}(u) = D_{33}(u) = [-a(u)]^L|0\rangle \\ &B_k(u)|0\rangle \neq 0 \qquad C_k(u) = 0 \qquad D_{ik}(u) = 0 \qquad (i \neq k, \quad i, k = 1, 2, 3). \end{aligned} \tag{30}$$

Substituting (29) into the Yang–Baxter algebra (6), we may deduce the following commutation relations:

$$D_{ab}(\mu)B_c(\lambda) = S_a\left[(-1)^{\epsilon_a\epsilon_b}\frac{r(\mu-\lambda)_{aa}^{bb}}{a(\mu-\lambda)}B_b(\lambda)D_{ac}(\mu) - (-1)^{\epsilon_a\epsilon_b}\frac{b(\mu-\lambda)}{a(\mu-\lambda)}B_b(\mu)D_{ac}(\lambda)\right]$$

$$A(\mu)B_c(\lambda) = S_c\left[\frac{1}{a(\lambda-\mu)}B_c(\lambda)A(\mu) - \frac{b(\lambda-\mu)}{a(\lambda-\mu)}B_c(\mu)A(\lambda)\right]$$

$$B_{a_1}(\lambda)B_{a_2}(\mu) = r(\lambda-\mu)_{a_1a_1}^{a_2a_2}B_{a_1}(\mu)B_{a_2}(\lambda)$$

where

$$r(u)_{aa}^{bb} = (b(u) + s_3a(u))I^{(1)}{}_{aa}^{bb} = \frac{1+s_3u}{1+s_1u}I^{(1)}{}_{aa}^{bb}.$$

Here, $I^{(1)}{}_{aa}^{bb}$ is the 9×9 identity matrix and $S_1 = \frac{1}{s_2}$, $S_2 = -1$, $S_3 = -1$. The eigenvalues of $\tau(u)$ read

$$\Lambda(u) = s_2^{-N_\downarrow}(-1)^{N_e}\prod_{j=1}^{n}\frac{1}{a(\lambda_j-u)} + [a(u)]^L s_2^{L-n}(-1)^{N_e+N_\downarrow}\prod_{j=1}^{n}\frac{1}{a(u-\lambda_j)}$$

with the following Bethe ansatz equations:

$$\left(\frac{1+s_1\lambda_k}{\lambda_k}\right)^L = s_2^{N_\downarrow}(-1)^{-N_e}s_2^{L-n}(-1)^{N_e+N_\downarrow}\prod_{\substack{k=1\\k\neq j}}^{N_e+N_\downarrow}\frac{s_1(\lambda_j-\lambda_k)+1}{s_1(\lambda_j-\lambda_k)-1}$$

and the energy is given by (20).

7. Algebraic Bethe ansatz for group 6

The final case to consider corresponds to the R-matrix

$$\check{R}(u) = 1 + u\Pi$$

$$= \begin{pmatrix}
1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & s_2u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & s_2u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+s_1u
\end{pmatrix}. \tag{31}$$

In contrast to the other cases considered, we choose the local vacuum state as $|0\rangle_j = (0,1,0,0)^t$. Acting the L-operator on this local vacuum state, we have

$$L_j(u)|0\rangle_j = \begin{pmatrix} s_2a(u) & 0 & 0 & 0\\ * & 1 & * & *\\ 0 & 0 & a(u) & 0\\ 0 & 0 & 0 & a(u)\end{pmatrix}|0\rangle_j. \tag{32}$$

Defining the vacuum state as $|0\rangle = \otimes_{j=1}^{L}|0\rangle_j$ we express the monodromy matrix as

$$T(u) = L_L(u)L_{L-1}(u)\cdots L_1(u) \equiv \begin{pmatrix} D_{11}(u) & C_1(u) & D_{12}(u) & D_{13}(u) \\ B_1(u) & A(u) & B_2(u) & B_2(u) \\ D_{21}(u) & C_2(u) & D_{22}(u) & D_{23}(u) \\ D_{31}(u) & C_3(u) & D_{32}(u) & D_{33}(u) \end{pmatrix} \tag{33}$$

and so the transfer matrix is

$$\tau(u) = D_{11}(u) + A(u) - D_{22}(u) - D_{33}(u).$$

The action of the monodromy matrix on the vacuum state is given by

$$\begin{array}{llll} D_{11}(u) = [s_2 a(u)]^L|0\rangle & A(u)|0\rangle = |0\rangle & D_{22}(u) = D_{33}(u) = [a(u)]^L|0\rangle & \\ B_k(u)|0\rangle \neq 0 & C_k(u) = 0 & D_{ik}(u) = 0 & (i \neq k, \quad i, k = 1, 2, 3). \end{array} \tag{34}$$

Substituting (33) into the Yang–Baxter algebra (6) we find that

$$D_{ab}(\mu)B_c(\lambda) = S_a\left[(-1)^{\epsilon_a\epsilon_b}\frac{r(\mu-\lambda)_{aa}^{bb}}{a^{(3)}(\mu-\lambda)}B_b(\lambda)D_{ac}(\mu) - (-1)^{\epsilon_a\epsilon_b}\frac{b^{(3)}(\mu-\lambda)}{a^{(3)}(\mu-\lambda)}B_b(\mu)D_{ac}(\lambda)\right]$$

$$A(\mu)B_c(\lambda) = S_c\left[\frac{1}{a^{(3)}(\lambda-\mu)}B_c(\lambda)A(\mu) - \frac{b^{(3)}(\lambda-\mu)}{a^{(3)}(\lambda-\mu)}B_c(\mu)A(\lambda)\right]$$

$$B_{a_1}(\lambda)B_{a_2}(\mu) = r(\lambda-\mu)_{a_1a_1}^{a_2a_2}B_{a_1}(\mu)B_{a_2}(\lambda)$$

where

$$r(u)_{aa}^{bb} = (b^{(3)}(u) + s_1 a^{(3)}(u))I^{(1)}{}_{aa}^{bb} = \frac{1+s_1u}{1+s_3u}I^{(1)}{}_{aa}^{bb}$$

and now $S_1 = \frac{1}{s_2}$, $S_2 = 1$, $S_3 = 1$. The eigenvalues for the transfer matrix read

$$\Lambda(u) = s_2^{-N_\uparrow}\prod_{j=1}^{L-N_1}\frac{1}{a^{(3)}(\lambda_j - u)} + [a(u)]^L s_2^{L-n}\prod_{j=1}^{L-N_1}\frac{1}{a^{(3)}(u-\lambda_j)}$$

so that the Bethe ansatz equations

$$\left(\frac{1+s_1\lambda_k}{\lambda_k}\right)^L = s_2^{N_\uparrow}s_2^{L-n}\prod_{\substack{k=1\\k\neq j}}^{L-N_1}\frac{s_3(\lambda_j-\lambda_k)+1}{s_3(\lambda_j-\lambda_k)-1}$$

are satisfied. Here the energy eigenvalue differs somewhat from the previous cases and has the form

$$E = \sum_{j=1}^{L-N_1}\frac{1}{\lambda_j(1+s_1\lambda_j)} - L.$$

8. Summary and discussion

In this paper, integrable extensions of the Hubbard model arising from supersymmetric group solutions, by means of the algebraic Bethe ansatz method, have been investigated. In particular, we have calculated explicitly the Bethe ansatz equations as well as the energy eigenvalues for six different classes of underlying R-matrices, which in fact correspond to 96 different possible physical Hamiltonians.

A natural direction for possible further research is to deal with physical applications of the above models. More specific future works will be: (i) studying low-energy behaviour and physical properties of the corresponding systems based on an analysis of the Bethe ansatz equations from these results, including: investigating the ground-state structure, computing the finite-size corrections to the low-lying energies, and calculating thermodynamic equilibrium properties, using the methods of Woynarovich [5]; (ii) employing some traditional mathematical methods such as the Wiener–Hopf technique to solve the special kind of integral equations arising from the thermodynamic Bethe ansatz equations, using the methods of Yang and Yang [28] and Babujian [29].

Acknowledgment

This work is supported by the Australian Research Council.

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